

PERIODICITY OF THE CAYLEY-DICKSON TWISTS

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ABSTRACT. Regarding the Cayley-Dickson algebras as twisted group algebras, this paper identifies the periodic character of these twists.

1. INTRODUCTION

The unit basis vectors $\{e_k\}$ of Cayley-Dickson algebras may be represented as a *twisted group* with e_0 as the group identity. For each of the eight Cayley-Dickson doubling products [4] there is a twisting map $\omega(p, q) : \mathbb{N}_0^2 \mapsto \{\pm 1\}$ (where \mathbb{N}_0 represents the non-negative integers) with the property that $e_p e_q = \omega(p, q) e_{p \oplus q}$ where \oplus is a group operation on \mathbb{N}_0 consisting of the ‘bit-wise exclusive or’ of the binary representations of non-negative integers.

This paper identifies the periodic nature common to all eight of these twisting maps.

2. BACKGROUND

Cayley-Dickson algebras are here regarded as twisted group algebras [9, 13] on 2^N dimensional Euclidean subspaces of the Hilbert space ℓ^p of square-summable sequences using the standard unit basis $e_0 = 1, 0, 0, 0, \dots$, $e_1 = 0, 1, 0, 0, 0, \dots$ etc. together with the standard norm and inner product. Since Cayley-Dickson algebras exist in a sequence $\{\mathbb{A}_k\}$ where the algebra \mathbb{A}_{N+1} consists of all ordered pairs $\{(a, b) | a, b \in \mathbb{A}_N\}$, an ordered pair (a, b) will be regarded as the ‘shuffle’ of sequence $a = a_0, a_1, a_2, \dots$ and sequence $b = b_0, b_1, b_2, \dots$ so that $(a, b) = a_0, b_0, a_1, b_1, \dots$. A real number x is identified with the sequence $x, 0, 0, 0, \dots$ so that $\mathbb{A}_0 = \mathbb{R}$. The *conjugate* x^* of a sequence $x = x_0, x_1, x_2, \dots$ satisfies $x + x^* \in \mathbb{R}$ thus we define $x^* = x_0, -x_1, -x_2, \dots$. Equivalently,

$$(2.1) \quad (a, b)^* = (a^*, -b)$$

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This approach generates the ‘shuffle basis’ on the infinite dimensional Cayley-Dickson algebra \mathbb{A} where the unit basis vectors are defined recursively as

$$(2.2) \quad e_0 = 1$$

$$(2.3) \quad e_{2k} = (e_k, 0) \text{ for } k \geq 0$$

$$(2.4) \quad e_{2k+1} = (0, e_k) \text{ for } k \geq 0$$

The algebra $\mathbb{A} = \bigcup \mathbb{A}_k$ contains as proper subalgebras the real numbers \mathbb{A}_0 , the complex numbers \mathbb{A}_1 , quaternions \mathbb{A}_2 , octonions \mathbb{A}_3 , sedenions \mathbb{A}_4 , etc. In general, if $a, b \in \mathbb{A}$ then so is the ordered pair (a, b) . Furthermore, for each $n \geq 0$, $\mathbb{A}_n \subset \mathbb{A}_{n+1}$.

There exist eight basic Cayley-Dickson *doubling products* [4] which are listed in Table 1.

$$\begin{aligned} P_0 &: (a, b)(c, d) = (ca - b^*d, da^* + bc) \\ P_1 &: (a, b)(c, d) = (ca - db^*, a^*d + cb) \\ P_2 &: (a, b)(c, d) = (ac - b^*d, da^* + bc) \\ P_3 &: (a, b)(c, d) = (ac - db^*, a^*d + cb) \\ P_0^\top &: (a, b)(c, d) = (ca - bd^*, ad + c^*b) \\ P_1^\top &: (a, b)(c, d) = (ca - d^*b, da + bc^*) \\ P_2^\top &: (a, b)(c, d) = (ac - bd^*, ad + c^*b) \\ P_3^\top &: (a, b)(c, d) = (ac - d^*b, da + bc^*) \end{aligned}$$

TABLE 1. The Eight Cayley-Dickson Doubling Products

A search of the literature on Cayley-Dickson algebras reveals the use of only the two doubling products P_3 and P_3^\top . In general for a doubling product P_k , $e_p e_q = e_r$ if and only if for the doubling product P_k^\top it is the case that $e_q e_p = e_r$. The basis product tables for P_k and P_k^\top are each the transpose of the other. Note, however, that this relationship holds only for the basis vectors. If $xy = z$ in P_k it does not follow that $yx = z$ for P_k^\top . Attention will be restricted in this paper to the first four of the products.

The proofs of the theorems involve the use of Cayley-Dickson ‘twist trees.’ [3] The next two sections discuss such twists and their trees.

3. CAYLEY-DICKSON TWISTS

For each of the eight Cayley-Dickson doubling products there is a unique *twist* function $\omega : \mathbb{N}_0 \times \mathbb{N}_0 \mapsto \{-1, 1\}$ such that for $p, q \in \mathbb{N}_0$

$$(3.1) \quad e_p e_q = \omega(p, q) e_{p \oplus q}$$

where $p \oplus q$ is the bit-wise ‘exclusive or’ of the binary representations of p and q . This relationship among the basis vectors is a natural result of regarding ordered pairs of sequences as the shuffle of the two sequences and is equivalent to addition in \mathbb{Z}_2^N .

To illustrate ,

$$5 \oplus 11 = 0101_B \oplus 1011_B = 1110_B = 14$$

so

$$e_5 e_{11} = \omega(5, 11) e_{14}$$

4. NAVIGATING THE TWIST TREES

Cayley-Dickson twist trees [3] are abstracted from observations about the structure of the multiplication table of the unit vectors e_k and are illustrated in Figures 1, 3 and 4.

To find the value of $\omega(p, q)$ in a basis vector product $e_p e_q = \omega(p, q) e_{p \oplus q}$ one needs a set of navigation instructions for the Cayley-Dickson twist tree. This set of instructions is symbolized by the bracketed ordered pair $[p; q]$ and details how to ‘navigate’ the tree by following a sequence of left-right instructions beginning at the root node. After navigating the tree according to the instructions $[p; q]$ the sign of the terminal node will be the value of $\omega(p, q)$.

The process of converting the symbol $[p; q]$ into a set of left-right navigation instructions for the tree is as follows:

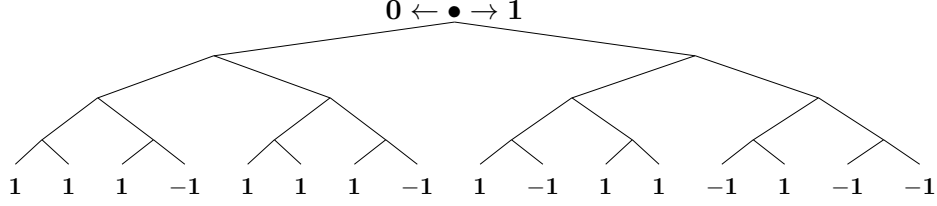
- (1) Convert $[p; q]$ to the binary representations of p and q padding the smaller of the two with leading 0’s when necessary to maintain an equal number of bits. Example: $[26; 42] = [011010; 101010]$.
- (2) Shuffle the two binary strings into binary doublets. $[011010; 101010] = 01, 10, 11, 00, 11, 00$
- (3) Interpret a 0 as an instruction to move down the left branch (L) from the current node and a 1 as an instruction to move down the right branch (R) from the current node. Example $01, 10, 11, 00, 11, 00 \Rightarrow LR, RL, RR, LL, RR, LL$.

The bracket notation $[p; q]$ denoting the shuffle of *binary numbers* p, q is used to avoid confusion with the parenthesis notation denoting the shuffle (x, y) of *number sequences* x, y .

Example. Refer for this example to the quaternion tree in Figure 1. The product $e_3 e_1 = \omega_3(3, 1) e_{3 \oplus 1} = \omega(3, 1) e_2$. The integers 3 and 1 are shuffled by pairing the bits of 3 with the bits of 1: $[3; 1] = [11; 01] = 10, 11$.

This string of bits is taken as the navigation instruction RL,RR for the twist tree. Following the instructions leads to the twelfth terminal node from the left which is labeled with 1. Thus $\omega(3, 1) = 1$. So $e_3 e_1 = e_2$. \triangle

If we identify e_1, e_2 and e_3 with quaternions i, j and k , respectively, this gives $ki = j$. This particular twist tree is identical for all four of the products P_0 through P_3 and produces the correct products for the quaternion basis vectors.

FIGURE 1. Quaternion Twist Tree P_0 through P_3 5. USING THE GENERAL TREE TO CALCULATE $e_p e_q$

A version of the general quaternion tree is developed in [4]. The general tree depicted in Figures 3 and 4 suffices for Cayley-Dickson algebras of any dimension for products P_0 through P_3 . A variation is valid for the transposes of these four products.

The meaning of the letters C, L, T, D and I are explained in [3] (although different letters were used there) but here it will suffice to say that they represent corner (C), left (L), top (T), diagonal (D) and interior (I) as illustrated in Figure 2 referencing sections of the product table of the basis vectors, specifically with regard to the values of $\omega(p, q)$ in those tables. Additionally, Figure 2 indicates how an ω table for \mathbb{A}_N transitions into an ω table for \mathbb{A}_{N+1} . For the complex numbers $\mathbb{C} = \mathbb{A}_1$, $C = T = L = D = I = 1$. The matrix on the right shows the twist matrix for the quaternions $\mathbb{Q} = \mathbb{A}_2$. Applying the transformation one more time would show the twist matrix for the octonions $\mathbb{O} = \mathbb{A}_3$. The Cayley-Dickson tree for the eight doubling products differ only in the behavior of their interior (I) nodes, requiring a separate I-tree for each of the eight.

$$\left(\begin{array}{c|c} C & T \\ \hline L & -D \end{array} \right) \Rightarrow \left(\begin{array}{cc|cc} C & T & T & T \\ L & -D & I & -I \\ \hline L & -I & -D & I \\ L & I & -I & -D \end{array} \right)$$

FIGURE 2. Matrix version of Figure 3

The twist tree in Figure 3 is valid for all four doubling products P_0 through P_3 but the trees for I shown in Figure 4 vary.

The following example illustrates how to use the general tree to find the product of two basis vectors using the product P_3 from Table 1.

Example. Let us calculate the product $e_{25}e_{17}$.

First find

$$25 \oplus 17 = 1101_B \oplus 1001_B = 0100_B = 8$$

Then

$$e_{25}e_{17} = \omega_3(25, 17)e_8$$

To find the tree navigation instructions shuffle the bits of the binary representations

$$[25; 17] = [1101; 1001] = 11, 10, 00, 11 (\Rightarrow RR, RL, LL, RR)$$

In Figure 3, 11, 10 takes us first to $-D$ then to $-I$.

Using the tree for I_3 in Table 4 but *reversing all the signs* since it's actually $-I_3$ in this case, the 00 takes us from $-I_3$ to $-I_3$ and the 11 takes us from $-I_3$ to $+I_3$. The sign of the final node is always the sign of $\omega(p, q)$.

Therefore, $\omega(25, 17) = 1$.

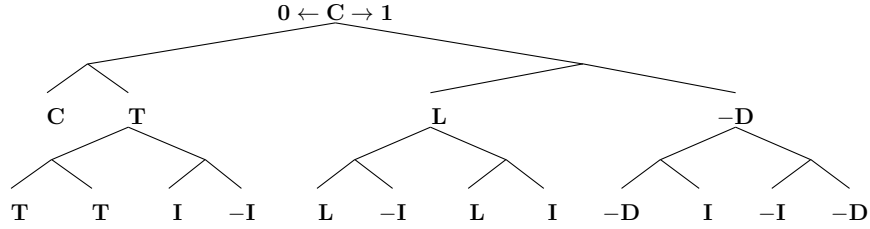


FIGURE 3. Twist tree for ω_0 through ω_3

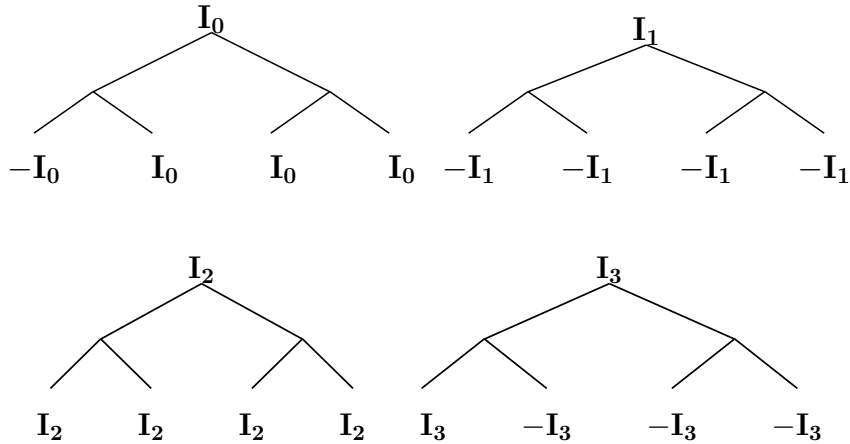


FIGURE 4. Twist trees for interior points of $\mathbb{N}_0 \times \mathbb{N}_0$

Thus $e_{25}e_{17} = e_8$.

△

6. PERIODIC PROPERTIES OF ω

second Because of the periodicities of ω , computing ω for large p, q can be simplified by computing $\omega(p', q')$ for smaller p', q' having the same ω value. This is useful, for example, in algorithms for computing products of vectors in higher dimensional Cayley-Dickson algebras. Investigation of applications of Cayley-Dickson algebras in areas such as signal processing, file compression or cryptography involving, for example \mathbb{A}_{10} would require 2^{20} computations of $\omega(p, q)$ when multiplying a single pair of vectors x and y .

Theorem 1. *The twists of all eight products satisfy the following: [4]*

$$\begin{aligned}\omega(p, 0) &= \omega(0, p) = 1 \text{ for all } p. \\ \omega(p, p) &= -1 \text{ for all } p > 0. \\ \omega(p, q) &= -\omega(q, p) \text{ for } 0 \neq p \neq q \neq 0\end{aligned}$$

The properties in Theorem 1 are well known but not theorem 2, which can be proved using the trees in Figures 3 and 4.

Remark. For a positive integer p the inequality $2^{N-1} \leq p < 2^N$ means that p cannot be expressed in binary form with fewer than N bits.

Theorem 2. *The first periodicity theorem: If $2^{N-1} \leq p < 2^N \leq q < 2^{N+1}$ and $k \geq 0$ then*

$$\omega(p, q) = \omega(p, q + k2^N)$$

Proof. Suppose $2^{N-1} \leq p < 2^N \leq q < 2^{N+1}$.

Then the two highest order bits of p are 01 and the two highest order bits of q are $1x$ where $x \in \{0, 1\}$.

So $[p; q] = [01 \cdots ; 1x \cdots] = 01, 1x, \cdots$. Thus, before applying the doublet $1x$ the current node is T.

Now consider $[p; q + k2^N] = 01, \cdots, 1x, \cdots$. The first ellipsis can contain only 00 or 01 so before applying the doublet $1x$ the node is also T.

Yet in both $[p; q] = 01, 1x, \cdots$ and $[p; q + k2^N] = 01, \cdots, 1x, \cdots$ the final ellipses are identical. Thus $\omega(p, q) = \omega(p, q + k2^N)$. □

Theorem 3. *The second periodicity theorem: If $2^{N-1} \leq p < 2^N$ and $2^{N-1} \leq q < 2^N$ and $k \geq 0$ then*

$$\omega(p, q) = \omega(p + k2^N, q + k2^N)$$

Proof. Here the path $[p; q] = [1 \cdots ; 1 \cdots] = 11, \cdots$ begins at $-D$.

In the path $[p + k2^N; q + k2^N] = [1 \cdots 1 \cdots ; 1 \cdots 1 \cdots] = 11, \cdots, 11, \cdots$ the first instructions brings one to $-D$ and the first ellipsis consists entirely

of 00s or 11s. Thus the path remains at $-D$ until reaching the final ellipsis. But in paths $[p; q]$ and $[p + k2^N; q + k2^N]$, the final ellipses are the same.

Thus $\omega(p, q) = \omega(p + k2^N, q + k2^N)$. \square

7. THE MODULARITY PROPERTIES

Using modular arithmetic, properties related to those in Theorems 2 and 3 may be stated.

Theorem 4. *If $2^{N-1} \leq p < 2^N \leq q$ then*

$$\omega(p, q) = \omega(p, 2^N + q \bmod 2^N)$$

Proof. Suppose $2^{M-1} \leq q < 2^M$ where $N \leq M$. Represent p by the binary string $0 \cdots 1 \cdots$ where the 1 is the N th bit from the right and 0 the M th. Represent q by the binary string $1 \cdots x \cdots$ where x is the N th bit from the right and 1 the M th. Then $[p; q] = 01, \dots, 1x, \dots$. In the tree diagram in Figure 3, on page 3 the instruction 01 brings us to T . The first ellipsis in $[p; q]$ is either null or consists of only 00 or 01, either of which leaves one at T . So the value of $\omega(p, q)$ will be the same as it would if the first ellipsis were empty. If q , with binary representation $1 \cdots x \cdots$ were to be replaced with q' with binary string $1x \cdots$ where the rightmost ellipsis of q is the same binary string as the ellipsis in q' , then $\omega(p, q) = \omega(p, q')$. So q may be replaced by $q' = 2^N + q \bmod 2^N$ and the value of ω will be the same. \square

Example. $\omega(5, 481) = \omega(5, 2^3 + 481 \bmod 2^3) = \omega(5, 9)$ \triangle

Theorem 5. *Suppose $2^{M-1} \leq p < 2^M$, $2^{M-1} \leq q < 2^M$ and $2^{N-1} \leq p \oplus q < 2^N$. Then $N < M$ and*

$$\omega(p, q) = \omega(2^N + p \bmod 2^N, 2^N + q \bmod 2^N)$$

Proof. Represent $p = 1 \cdots x \cdots$ and $q = 1 \cdots y \cdots$ with both 1s the M th bit from the right and with x and y the N th bits from the right. Then $[p; q] = 11, \dots xy, \dots$ with the first ellipsis being empty or consisting of only 11 or 00. So applying these navigating instructions for the tree in Figure 3 places one at $-D$ when arriving at the instruction xy . So it would have been the same as with binary $p' = 1x \cdots$ and $q' = 1y \cdots$ with the 1s occupying the $(N+1)$ st bit from the right (representing 2^N) and the two ellipses the same as the rightmost ellipses of p and q . Thus $\omega(p, q) = \omega(p', q')$ where $p' = 2^N + p \bmod 2^N$ and $q' = 2^N + q \bmod 2^N$. \square

Example. $483 \oplus 481 = 2$ and $2^1 \leq 2 < 2^2$ so $N = 2$. So one concludes that $\omega(483, 481) = \omega(2^2 + 483 \bmod 2^2, 2^2 + 481 \bmod 2^2) = \omega(7, 5)$ \triangle

8. CONCLUSION

The lack of a standard indexing system for the Cayley-Dickson basis vectors together with the fact that there is more than one possible doubling

product have tended to obscure the basic periodicity of the twisting maps of these products. It is hoped that these modest results will help to clarify the issue and prove useful for further research.

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